teachyourselfmath issue 1

EXPLAINER

Division algorithm. For any integers a and $b > 0$ there exist unique integers q, r with

$$a = qb + r, \quad 0 \leq r < b$$

This lets you reason about all integers by finitely many "cases mod b" (e.g., any integer is $0, 1, 2$ mod 3).

Divisibility & linear combos. Write $a \mid b$ when $b = ac$ for some integer c. If $a \mid b$ and $a \mid c$, then $a \mid (bx + cy)$ for any integers x, y.

Greatest common divisor (gcd). For integers a, b not both zero, $\gcd(a, b)$ is the largest positive integer dividing both. A key fact (Bézout):

$$\gcd(a, b) = ax + by \quad \text{for some integers } x, y$$

Two integers are **coprime** if their gcd is 1.

Euclid's algorithm. Repeatedly apply the division algorithm to get remainders decreasing to 0; the last nonzero remainder is $\gcd(a, b)$. Back-substitute to express the gcd as $ax + by$.

Euclid's lemma. If $\gcd(a, b) = 1$ and $a \mid bc$, then $a \mid c$. (Multiply $ax + by = 1$ by c.)

Least common multiple (lcm). For $a, b > 0$,

$$\gcd(a, b) \cdot \operatorname{lcm}(a, b) = ab$$

Linear Diophantine equations. For $ax + by = c$, solutions in integers exist iff $\gcd(a, b) \mid c$. If (x_0, y_0) is one solution and $d = \gcd(a, b)$, all solutions are

$$x = x_0 + \tfrac{b}{d}t, \quad y = y_0 - \tfrac{a}{d}t, \quad t \in \mathbb{Z}$$

EASY STUFF

1) Use the division algorithm to write $2025 = 37q + r$ with $0 \leq r < 37$.
2) Show the square of any odd integer has the form $8k + 1$.
3) Compute $\gcd(252, 198)$ by Euclid's algorithm, then find integers x, y with $252x + 198y = \gcd(252, 198)$.
4) Solve in integers: $15x + 21y = 6$. Give the general solution.

sketches

1) $37 \cdot 54 = 1998$, so $2025 = 37 \cdot 54 + 27$.
2) Any integer is $4q, 4q + 1, 4q + 2, 4q + 3$. Odd $\Rightarrow 4q + 1$ or $4q + 3$; squaring gives $16q^2 + 8q + 1 = 8(2q^2 + q) + 1$.
3) $252 = 1 \cdot 198 + 54$, $198 = 3 \cdot 54 + 36$, $54 = 1 \cdot 36 + 18$, $36 = 2 \cdot 18 \Rightarrow \gcd = 18$. Back-substitute: $18 = 4 \cdot 54 - 198 = 4(252 - 198) - 198 = 4 \cdot 252 - 5 \cdot 198$.
4) $\gcd(15, 21) = 3 \mid 6$. One solution from $21 - 15 = 6$: $x = -1, y = 1$. General: $x = -1 + 7t,\ y = 1 - 5t,\ t \in \mathbb{Z}$.

MEDIUM STUFF

1) Prove any square is $3k$ or $3k + 1$.
2) Show any cube is $9k, 9k + 1$, or $9k + 8$.
3) Prove $3a^2 - 1$ is never a perfect square.
4) If $a = qb + r$, prove $\gcd(a, b) = \gcd(b, r)$.
5) Prove Euclid's lemma: if $\gcd(a, b) = 1$ and $a \mid bc$, then $a \mid c$.
6) Compute $\gcd(84, 330)$ and $\mathrm{lcm}(84, 330)$.
7) Solve $18x + 5y = 48$ in integers; list all positive solutions.
8) Solve $54x + 21y = 906$ in integers; give one small positive solution and the general form.
9) Prove: if $d = \gcd(a, b)$ then $\gcd(a/d, b/d) = 1$.
10) Show no base-10 repunit $11 \ldots 1$ is a perfect square.

sketches

1) Cases $n = 3q, 3q + 1, 3q + 2$ give $9q^2, 9q^2 + 6q + 1, 9q^2 + 12q + 4 \equiv 0, 1, 1 \pmod 3$.
2) Cases mod 9: cubes are $\equiv 0, \pm 1 \pmod 9$.

3) From (1), $a^2 \equiv 0, 1 \pmod 3 \Rightarrow 3a^2 - 1 \equiv -1, 2 \pmod 3$. Squares are $0, 1 \pmod 3$, contradiction.

4) Common divisors of a, b are exactly common divisors of $b, r = a - qb$. Maximal ones match.

5) With $\gcd(a, b) = 1$, take $ax + by = 1$. Multiply by c: $c = a(cx) + b(cy)$. If $a \mid bc$, then $a \mid c$.

6) Euclid gives $\gcd(84, 330) = 6$. Then $\text{lcm} = \frac{84 \cdot 330}{6} = 4620$.

7) Mod 5: $18x \equiv 48 \equiv 3 \Rightarrow 3x \equiv 3 \Rightarrow x \equiv 1 \pmod 5$. Take $x = 1$, then $y = 6$.
   General: $x = 1 + 5t,\ y = 6 - 18t$. Positivity forces $t = 0 \Rightarrow (1, 6)$.

8) $\gcd(54, 21) = 3 \mid 906 \Rightarrow$ reduce: $18x + 7y = 302$. Mod 7: $18x \equiv 302 \equiv 1 \Rightarrow 4x \equiv 1 \Rightarrow x \equiv 2 \pmod 7$.
   Take $x = 2 \Rightarrow y = 38$. General: $x = 2 + 7t,\ y = 38 - 18t$.

9) From $d = ax_0 + by_0$, divide by d: $1 = (a/d)x_0 + (b/d)y_0 \Rightarrow$ coprime.

10) Write $R_n = 11\ldots1 = (10^n - 1)/9$. Note $R_n \equiv 3 \pmod 4$.
    Squares are $0, 1 \pmod 4$, so R_n is not a square.

HARD STUFF

Find the positive integers x, y minimizing $x + y$ subject to $158x - 57y = 7$. Also give the full solution set.

sketch. $\gcd(158, 57) = 1$. Extended Euclid yields $1 = -22 \cdot 158 + 61 \cdot 57$.
Multiply by 7: $7 = (-154) \cdot 158 + 427 \cdot 57$, so a particular solution is $(x_0, y_0) = (-154, -427)$.
General solution:

$$x = -154 - 57t, \quad y = -427 - 158t, \quad t \in \mathbb{Z}$$

Set $t = -k$: $x = -154 + 57k,\ y = -427 + 158k$.
Positivity needs $k \geq 3$. For $k = 3$: $x = 17,\ y = 47$, $x + y = 64$.
Larger k increases both, so the minimum is $(17, 47)$.
Full positive solutions: $x = 17 + 57m,\ y = 47 + 158m,\ m \geq 0$.

"Mathematics is the queen of the sciences, and number theory is the queen of mathematics." — *Carl Friedrich Gauss*